Simplification of Gauss's third Proof that every Algebraic Equation has a Root.

By Maxime Bôcher.

While Gauss's first and second proofs of the fundamental theorem of algebra have found many commentators, some of whom have even succeeded in essentially simplifying them, Gauss's third proof (Ges. Werke, p. 59 and p. 107), which was evidently considered by the author himself as not the least worthy of notice, is seldom referred to and, as far as I am aware, no attempt has been made to simplify it. This last fact is doubtless due to the very simplicity of the original proof which precludes any very great simplification, while its apparent failure to excite the interest of mathematicians may probably be in part explained by the fact that it appears at first sight to consist of nothing more than a skillful manipulation of formulæ. I have shown on another occasion* that Gauss's proof amounts practically to the application of a familiar theorem in the theory of the potential to the real part of the function $zf'(z)/f(z)$, where $f(z)=0$ is the equation for which we wish to prove the existence of a root. The simplified proof which follows is really equivalent to the application of the same method to the function $1/f(z)$. I have, however, followed as closely as possible the form of Gauss's proof.

We will write the equation for which we wish to prove the existence of a root in the form

$$a_0z^n + a_1z^{n-1} + \ldots\ldots + a_{n-1}z + 1 = 0,$$

and we will suppose that the coefficients $a_0, a_1, \ldots\ldots a_{n-1}$ are real.† Let us

* In a note recently sent to the Bulletin of the American Mathematical Society.

† Both the proof here given and Gauss's proof could easily be extended so as to cover the case of complex coefficients.

write $z = r(\cos\phi + i\sin\phi)$. Then the first member of the equation may be written $t + ui$ where

$$t = a_0 r^n \cos n\phi + a_1 r^{n-1}\cos(n-1)\phi + \ldots\ldots + a_{n-1} r\cos\phi + 1,$$
$$u = a_0 r^n \sin n\phi + a_1 r^{n-1}\sin(n-1)\phi + \ldots\ldots + a_{n-1} r\sin\phi.$$

We have to prove that for some pair of values of r and ϕ, t and u both vanish. Let us consider the double integral

$$\Omega = \int_{r=0}^{r=R}\int_{\phi=0}^{\phi=2\pi} \frac{(u^2 - t^2)\,t' - 2tuu'}{r\,(t^2 + u^2)^2}\,dr\,d\phi = \int_{r=0}^{r=R}\int_{\phi=0}^{\phi=2\pi} y\,.\,dr\,.\,d\phi,$$

in which

$$t' = a_0 n r^n \cos n\phi + a_1(n-1)\,r^{n-1}\cos(n-1)\phi + \ldots\ldots + a_{n-1} r\cos\phi,$$
$$u' = a_0 n r^n \sin n\phi + a_1(n-1)\,r^{n-1}\sin(n-1)\phi + \ldots\ldots + a_{n-1} r\sin\phi.$$

The function y which stands above under the signs of integration is evidently single-valued and continuous for all values of r and ϕ except such as make $t^2 + u^2 = 0$, i. e. such as make t and u vanish together, since the factor r which appears in the denominator occurs also in each term of the numerator. If, then, there were no values of r and ϕ for which t and u vanish together, we could, in computing the value of Ω, perform the two integrations in either order.

We have the following indefinite integral formulæ, as is seen by direct differentiation:

$$\int y\,dr = \frac{t}{t^2 + u^2}, \quad \int y\,d\phi = \frac{-u}{r\,(t^2 + u^2)}.$$

If we take this last integral between the limits $\phi = 0$ and $\phi = 2\pi$, we evidently get the value zero, since the indefinite integral vanishes at both limits. Thus by integrating first with regard to ϕ and then with regard to r, we get

$$\Omega = 0.$$

Let us now integrate first with regard to r and then with regard to ϕ. We get, by using the first indefinite integral given above,

$$\Omega = \int_{\phi=0}^{\phi=2\pi}\left[\frac{T}{T^2 + U^2} - 1\right] d\phi,$$

where T and U are the values of t and u when $r = R$. We will now take R,

which has so far been entirely undetermined, greater than the largest of the following positive quantities:

$$\frac{\sqrt{8}\,n\,|a_1|}{|a_0|},\quad \sqrt{\frac{\sqrt{8}\,n\,|a_2|}{|a_0|}},\quad \sqrt[3]{\frac{\sqrt{8}\,n\,|a_3|}{|a_0|}},\;\ldots\; \sqrt[n]{\frac{\sqrt{8}\,n}{|a_0|}}.^{*}$$

It is easily seen that for every value of ϕ at least one of the following inequalities will hold:

$$|T|>n,\quad |U|>n,$$

viz. the first when $|\cos n\phi|\geqq\sqrt{\tfrac{1}{2}}$, the second when $|\sin n\phi|\geqq\sqrt{\tfrac{1}{2}}$. To prove this let us indicate by ε_p, ε_p', η_p quantities numerically less than 1 except that $|\eta_1|\leqq 1$. Then we can write, when $|\cos n\phi|\geqq\sqrt{\tfrac{1}{2}}$,

$$R=\frac{\sqrt{8}\,n\,|a_1|}{|a_0|\,\varepsilon_1},\quad R^2=\frac{\sqrt{8}\,n\,|a_2|}{|a_0|\,\varepsilon_2},\;\ldots\; R^n=\frac{\sqrt{8}\,n}{|a_0|\,\varepsilon_n},$$

$$T=\frac{\sqrt{8}\,n}{|a_0|\,\varepsilon_n}\left[a_0\cos n\phi+\frac{|a_0|\,\varepsilon_1'}{\sqrt{8}\,n}+\frac{|a_0|\,\varepsilon_2'}{\sqrt{8}\,n}+\ldots+\frac{|a_0|\,\varepsilon_n}{\sqrt{8}\,n}\right]$$

$$=\frac{\sqrt{8}\,n}{|a_0|\,\varepsilon_n}\left[\frac{|a_0|}{\sqrt{2}\,\eta_1}+\frac{|a_0|\,\eta_2}{2\sqrt{2}}\right]=\frac{2n}{\varepsilon_n}\left[\frac{1}{\eta_1}+\frac{\eta_2}{2}\right]=\frac{n}{\varepsilon_n\eta_3}.$$

In the same way it can be proved that when $|\sin n\phi|\geqq\sqrt{\tfrac{1}{2}}$, $|U|>n$.

This being the case,

$$\frac{T}{T^2+U^2}-1=-\frac{T^2+U^2-T}{T^2+U^2}$$

will certainly be negative for all values of ϕ, and its integral from $\phi=0$ to $\phi=2\pi$ will be negative. Our hypothesis that there is no pair of values of r and ϕ for which t and u both vanish thus leads us to the contradiction that Ω has the value zero when computed in one way and a negative value when computed in another way. Our original equation must therefore have a root.

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* $|a|$ means the absolute (numerical) value of a. We might easily have found smaller upper limits for R had there been an object in doing so.